Social Choice Theory

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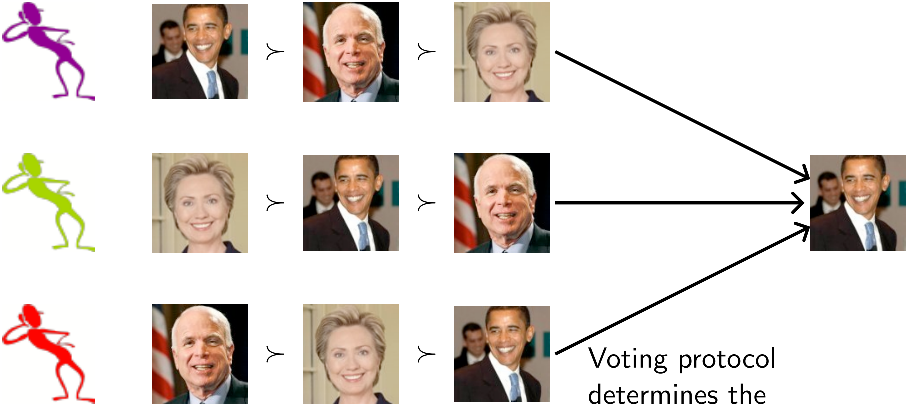
Introduction

- Social choice: preference aggregation
- Our settings
  - A set of agents have preferences over a set of alternatives
  - Taking preferences of all agents, the mechanism outputs a social preference over the set of alternatives or output a single winner
  - Hope to satisfy some desired properties
- Voting protocols are examples of social choice mechanisms
- Readings: SLB 9.1 – 9.4

Voting



Voting protocol determines the winner or the final ordering

Example Voting Protocols

- Plurality Voting
  - Each voter cast a single vote.
  - The candidate with the most votes is selected.
- Approval Voting
  - Each voter can cast a single vote for as many candidates as he wants.
  - The candidate with the most votes is selected.
- Single Transferable Vote (Instant Roundoff)
  - Each candidate votes for their most-preferred candidate
  - The candidate with the fewest votes is eliminated
  - Each voter who voted for the eliminated candidate transfers their vote to their most-preferred candidate among the remaining candidates
- Borda Voting
  - Each voter submits a full ordering on the m candidates
  - Candidates of an ordering get score $(m-1, m-2, ..., 0)$
  - The candidate with the highest score is selected

Pairwise Elections

2 prefer Obama to McCain

2 prefer McCain to Hillary

2 prefer Obama to Hillary

More Voting Protocols

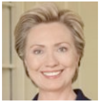
- Pairwise elimination
  - Pair candidates with a schedule
  - The candidate who is preferred by a minority of voters is deleted
  - Repeat until only one candidate is left
- Slater
  - The overall ordering that is inconsistent with as few pairwise elections as possible is selected.
  - NP-hard
- Kemeney
  - The overall ordering that is inconsistent with as few votes on pairs of candidates as possible.
  - NP-hard
- ... and many other voting rules

What is the perfect voting protocol?

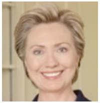
Condorcet Condition

- A candidate is a Condorcet winner if it wins all its pairwise elections.
- A voting protocol satisfies the Condorcet condition, if the Condorcet winner, if exists, must be elected by the protocol.
- Condorcet winner may not exist.
- Many voting protocols do not satisfy the Condorcet condition.

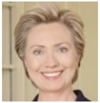
Condorcet Circle

  $\succ$  $\succ$ 

2 prefer Obama to McCain

  $\succ$  $\succ$ 

2 prefer McCain to Hillary

  $\succ$  $\succ$ 

2 prefer Hillary to Obama

?

An Example of Condorcet Condition

499 agents: $a \succ b \succ c$
3 agents: $b \succ c \succ a$
498 agents: $c \succ b \succ a$

- Which candidate is the Condorcet winner if exists?
- Which candidate is the plurality voting selected?
- Which candidate is the Single Transferable Vote selected?

Voting Paradox: Sensitivity to A Losing Candidate

35 agents: $a \succ c \succ b$
33 agents: $b \succ a \succ c$
32 agents: $c \succ b \succ a$

- Which alternative is the winner under plurality voting?
- Which alternative is the winner under Borda voting?
- What happens if c drops off?

Notations

- N: a set of individuals, $|N| = n$
- A: a set of alternatives, $|A| = m$
- $\succ_i$: agent i's preference over A (e.g. $a_i \succ_i a_3 \succ_i a_5$)
- L: the set of total orders, $\succ \in L$
- L^n: the set of preference profiles, $[\succ] \in L^n$
- A social welfare function is a function $W : L^n \rightarrow L$
- $\succ_W$: the preference ordering selected by W
- A social choice function is a function $C : L^n \rightarrow A$

Social Welfare Function: Pareto Efficiency

- A social welfare function W is Pareto efficient if for any a_1, $a_2 \in A$, $\forall a_1 \succ_i a_2$ implies that $a_1 \succ_W a_2$.
- It means that when all agents agree on the ordering of two alternatives, the social welfare function must select the ordering.

Social Welfare Function: Independence of Irrelevant Alternatives (IIA)

- A social welfare function W is independent of irrelevant alternatives if, for any a_1, $a_2 \in A$ and any two preference profiles $[\succ']$, $[\succ''] \in L^n$, $\forall i$

$$(a_1 \succ'_i a_2 \text{ if and only if } a_1 \succ''_i a_2) \Rightarrow$$
$$(a_1 \succ_{W([\succ'])} a_2 \text{ if and only if } a_1 \succ_{W([\succ''])} a_2).$$

- IIA means that if (1) W ranks a_1 ahead of a_2 now, and (2) we change the preferences without change the relative preferences between a_1 and a_2, then a_1 is still ranked ahead of a_2.
- An example with plurality voting protocol

499 agents:	$a \succ b \succ c$		$a \succ b \succ c$
3 agents:	$b \succ c \succ a$	$\Rightarrow$	$b \succ c \succ a$
498 agents:	$c \succ b \succ a$		$b \succ a \succ c$

- None of our rules satisfy IIA

Social Welfare Function: Nondictatorship

- We do not have a dictator if there does not exist an i such that $\forall a_1, a_2,$

$$a_1 \succ_i a_2 \Rightarrow a_1 \succ_W a_2$$

- Nondictatorship means that there does not exist a voter such that the social welfare function W always output the voter's preference

Arrow's Impossibility Results (1951)

- If $|A| \geq 3$, any social welfare function W can not simultaneously satisfy
  - Pareto efficiency
  - Independence of irrelevant alternatives
  - Nondictatorship
- Most influential result in social choice theory
- Read the proof

Maybe asking for a complete ordering is too much? Let's consider social choice functions.

Social Choice Function: Weak Pareto Efficiency

- A social choice function C is weakly Pareto efficient if for any preference profile $[\succ] \in L^n$, if there exist a pair of alternatives a_1 and a_2 such that $\forall i \in N$, $a_1 \succ_i a_2$, then $C(\succ) \neq a_2$.

- It means that a dominated alternative can not be selected.

- Weak Pareto efficiency implies unanimity: If a_1 is the top choice for all agents, we must have $C[\succ] = a_1$.

- Pareto efficient rules satisfy week Pareto efficiency. But the reverse is not true.

Social Choice Function: Strong Monotonicity

- A social choice function C is strongly monotonic, if for any preference profile $[\succ]$ with $C[\succ] = a$, then for any other preference profile $[\succ']$ with the property that

$$\forall i \in N, \forall a' \in A, a \succ'_i a' \text{ if } a \succ_i a',$$

it must be that $C[\succ'] = a$.

- Strong monotonicity means that if
  - The current winner is a
  - We change the preference profile in the way such that for if alternative a' ranks below a previously it is still below a in the new preference

  Then, a is the winner for the new preference profile.
- An example with STV

9 agents:	$a \succ b \succ c$		12 agents:	$a \succ b \succ c$
9 agents:	$b \succ c \succ a$	$\Rightarrow$	6 agents:	$b \succ c \succ a$
7 agents:	$c \succ a \succ b$		7 agents:	$c \succ a \succ b$

- None of our rules satisfy strong monotonicity

Social Choice Function: Nondictatorship

- A social choice function C is nondictatorial if there does not exist an agent i such that C always outputs the top choice of i.

Muller-Satterthwaite's Impossibility Results (1977)

- If $|A| \geq 3$, any social choice function C can not simultaneously satisfy
  - Weak Pareto efficienty (unanimity)
  - Strong monotonicity
  - Nondictatorship
- Social choice functions are no simpler than social welfare functions
- Intuition: We can repeatedly probe a social choice function for given pairs of alternatives, and then construct a full social welfare ordering.

Social Choice Function: Manipulability

- A social choice function is manipulable if some voter can be better off by lying about his preference
- An example with plurality voting

$$\begin{aligned} \text{1 agent: } & a \succ b \succ c \\ \text{2 agents: } & b \succ c \succ a \\ \text{2 agents: } & c \succ b \succ a \end{aligned}$$

Social Choice Function: Onto

- A social choice function C is onto if for each $a \in A$ there is a preference profile $[\succ] \in L^n$ such that $C([\succ]) = a$.
- Onto means that every alternative can be a winner under some preference profile.

Gibbard-Satterthwaite's Impossibility Results (1973, 1975)

- If $|A| \geq 3$, any social choice function can not simultaneously satisfy
  - Nonmanipulable
  - Onto
  - Nondictatorship

What's possible?

Some Possibility Results: Single-Peaked Preferences

- Alternatives are a linear order (e.g. ordered on real line)
- Single-peaked preference: every voter has his most-preferred alternative and prefers alternatives that are closer to his favorite alternative
- Ask the voters to only report his favorite alternative
- The social choice function chooses the median voter's favorite alternative as the winner
- The winner is a Condorcet winner
- Nonmanipulable!