CS286r Iterative Combinatorial Exchanges
Homework 2: Mechanism Design

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Due: Tuesday 2/24/2004, in the beginning of class. You may use any sources that you want, but you must cite the sources that you use. You can also work in a group, just list off the people you're working with. **If you took the class last year please turn in a brief paper review on one of the papers that will be distributed later this week instead.** Please work hard on making the proofs clear, concise, and easy to read.

1. Consider a problem in which the outcome space, $\mathcal{O} \subset \mathbb{R}$, and each agent i, with type θ_i, has *single-peaked* preferences, $u_i(o, \theta_i)$ over outcomes. In particular, each agent, i, with type θ_i, has a *peak*, $p_i(\theta_i) \in \mathcal{O}$, such that $p(\theta_i) \geq d > d'$ or $d' > d \geq p(\theta_i)$ imply that $u_i(d, \theta_i) > u_i(d', \theta_i)$ (p.10–11, M.Jackson "Mechanism Theory" handout).

   (a) (10 pts) Show that the "median selection" mechanism, in which each agent declares its peak and the mechanism selects the median (with a tie break in the case of an even number of agents) is *strategyproof*, and implements a *Pareto Optimal* outcome.

   (b) (5 pts) Let N denote the number of agents. Suppose, in addition, that the mechanism can position its own $N-1$ "phantom peaks", before the peaks from the agents are received. Show that the median selection mechanism applied to the combined, $2N-1$, peaks remains strategyproof.

   (c) (5 pts) In combination with the phantom peaks, the median selection mechanism can implement a rich variety of outcomes. Describe a method to position the peaks to implement the kth order statistic *of the peaks announced by agents,* for some $1 \leq k \leq N$. (i.e. implement the outcome at the kth largest peak)

2. Consider the design of a mechanism for a simple bilateral trading problem, in which there is a single seller (agent 1), with a single item, and a single buyer (agent 2). The outcome of the mechanism defines an *allocation*, (x_1, x_2), where $x_i \in \{0, 1\}$ and $x_i = 1$ if agent i receives the item in the allocation, and defines *payments* (p_1, p_2) by the agents to the mechanism. Let v_i denote the value of

agent i for the item, and suppose quasilinear preferences, such that $u_i(x_i, p_i) = x_i v_i - p_i$ is the utility of agent i for outcome (x_1, x_2, p_1, p_2).

(a) (10 pts) Specify the Vickrey-Clarke-Groves mechanism for the problem; i.e. define the strategy space, the rule to select the allocation based on agent strategies, and the rule to select the payments based on agent strategies.

(b) (5 pts) Provide a simple example to show that the VCG mechanism for the exchange is not (*ex post*) weak budget-balanced.

(c) (5 pts) Is it possible to build an exchange mechanism that leads to an efficient allocation in a *dominant strategy* equilibrium, and is also *ex post* weak budget-balanced and *interim* individual-rational? What about in *Bayes-Nash* equilibrium? [**Hint:** Either refer to the appropriate impossibility theorem, or describe in brief terms the appropriate mechanism.]

3. (10 pts) Show that if $f : \Theta \rightarrow \mathcal{O}$ is truthfully implementable in dominant strategies when the set of possible types is Θ_i for $i = 1, \ldots, N$ [i.e. the direct revelation mechanism, $\mathcal{M} = (\Theta, f)$, is strategyproof], then when each agent i's set of possible types is $\hat{\Theta}_i \subset \Theta_i$ (for $i = 1, \ldots, N$) the social choice function $\hat{f} : \hat{\Theta} \rightarrow \mathcal{O}$ satisfying $\hat{f}(\theta) = f(\theta)$ for all $\theta \in \hat{\Theta}$ is truthfully implementable in dominant strategies.

4. (10 pts) Consider a problem in which the mechanism must make a choice $k \in \mathcal{K}$, and agents have all possible preference orderings across outcomes. Let $a \succ_i b$, for $a, b \in \mathcal{K}$ denote a preference type in which agent i prefers a to b. There are at least three agents. Explain (from first principles) why the following social-choice function cannot be implemented in a dominant-strategy equilibrium by any mechanism:

$$f(\theta) = \begin{cases} a & \text{, if for all } i \text{ we have } a \succ_i b \text{ for all } b \neq a \\ a^* & \text{, otherwise.} \end{cases}$$

where θ denotes the preferences of agents and a^* is an arbitrary member of $\mathcal{K}$.