CS286r Iterative Combinatorial Exchanges Homework 3: A Little Mech. Design and a Slither of LP

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Due: Thrusday 3/4/2004, in the beginning of class. You may use any sources that you want, but you must cite the sources that you use. You can also work in a group, just list off the people you're working with. **If you took the class last year please turn in a brief paper review on one of the papers that will be distributed later this week instead.** Please work hard on making the proofs clear, concise, and easy to read.

1. (30 pts) Consider a *double auction* (DA), with m buyers and n sellers, each trading a single item. Buyers and sellers submit bids and asks, and the DA determines the trade, and agents' payments. Let $b_1, \ldots, b_m$ denote the bid prices from buyers, and assume $b_1 \geq b_2 \geq \ldots \geq b_m \geq 0$. Let $s_1, \ldots, s_n$ denote the ask prices from sellers, and assume $0 \leq s_1 \leq s_2 \leq \ldots s_n$. In addition, define $b_{m+1} = 0$ and $s_{n+1} = \infty$. Later we refer to the following examples: (i) buyer values $9, 8, 7, 4$, seller values $2, 3, 4, 5$; (ii) buyer values $9, 8, 7, 4$, seller values $2, 3, 4, 12$.

   (a) (10 pts) Define the VCG mechanism for this problem, and show that the mechanism is not *ex post* weak BB for example (i). [**Hint:** it is useful to interpret a bid, or an ask, as an agent's claim about its value for the item. Define the trades implemented, payment by each buyer, payment to each seller.]

   Consider the following modified trading mechanism, the *McAfee-DA*:

   (1) select k, s.t. $b_k \geq s_k$ and $b_{k+1} < s_{k+1}$.

   (2) compute *candidate trading price*, $p_0 = 1/2(b_{k+1} + s_{k+1})$.

   (3) if $s_k \leq p_0 \leq b_k$, then the buyers/sellers from 1 to k trade at price p_0; otherwise, the buyers/sellers from 1 to $k-1$ trade, and each buyer pays b_k, each seller gets s_k.

   (b) (15 pts) Prove that the McAfee-DA is strategy-proof, and *ex post* weak budget-balanced.

(c) (5 pts) Run the McAfee-DA on examples (i) and (ii). Is the DA efficient?

2. (5 pts) Formulate the following as linear programs:

   (i) $u = \min\{x_1, x_2\}$, assuming that $0 \leq x_j$ for $j = 1, 2$

   (ii) $v = |x_1 - x_2|$, assuming that $0 \leq x_j$, for $j = 1, 2$

3. (10 pts)

   (i) Show that

$$
\begin{aligned}
X &= \{x \in \{0,1\}^4 \ : \ 93x_1 + 49x_2 + 37x_3 + 29x_4 \leq 111\} \\
&= \{x \in \{0,1\}^4 \ : \ 2x_1 + x_2 + x_3 + x_4 \leq 2\} \\
&= \{x \in \{0,1\}^4 \ : \ 2x_1 + x_2 + x_3 + x_4 \leq 2; \ x_1 + x_2 \leq 1; \ x_1 + x_3 \leq 1; \ x_1 + x_4 \leq 1\}
\end{aligned}
$$

   [**Hint:** first enumerate the feasible solutions defined for the first formulation.]

   Now consider solving the problem $\max\{c^T x \ : \ x \in X\}$ as a linear program, i.e. with $x \in \{0,1\}^4$ replaced by $x \in \mathbb{R}^4_+$.

   (ii) Will the LP relaxation of the integer program provide an upper- or lower-bound?

   (iii) Which formulation of the contraints would you expect to provide the tightest bound? Why?

4. (20 pts) John Harvard is attending a summer school where he must take four courses per day. Each course lasts an hour, but because of the large number of students, each course is repeated several times per day by different teachers. Section i of course k denoted (i, k) meets at the hour t_{ik}, where courses start on the hour between 10am and 7pm. Let T denote the set of start times, and suppose there are m courses, and that each course has n sections. John's preferences for when he takes courses are influenced by the reputation of the teacher, and also the time of day. Let v_{ik} be his value for section (i, k).

   (i) Carefully formulate an integer program to choose a feasible course schedule that maximizes the sum of John's preferences.

   (ii) Modify the formulation in (i), so that John never has more than two consecutive hours of classes without a break.

   (iii) Modify the formulation in (i), so that John chooses the schedule in which he starts his day as late as possible (ignoring the v_{ik} preferences). [**Hint:** you'll need to introduce a new decision variable.]

   (iv) How might you formulate the problem to select the latest schedule that also maximizes John's preferences?