Truthful Fair Division

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Abstract

We address the problem of fair division, or cake cutting, with the goal of finding truthful mechanisms. In the case of a general measure space ("cake") and non-atomic, additive individual preference measures - or utilities - we show that there exists a truthful "mechanism" which ensures that each of the k players gets at least $1/k$ of the cake. This mechanism also minimizes risk for truthful players. Furthermore, in the case where there exist at least two different measures we present a different truthful mechanism which ensures that each of the players gets more than $1/k$ of the cake.

We then turn our attention to partitions of indivisible goods with bounded utilities and a large number of goods. Here we provide similar mechanisms, but with slightly weaker guarantees. These guarantees converge to those obtained in the non-atomic case as the number of goods goes to infinity.

1 Introduction

The basic setting of fair partition problems includes a "cake" - some divisible resource - and a number of players, each with different preferences with regards to the different "pieces" of the cake. The task is to divide the cake among the players in a way that would be "fair" according to some fairness notion. This applies to numerous situations, from divorce trials to nuclear arms reduction, see e.g. [1].

This problem has been studied extensively both in the non-atomic case and in the case of indivisible goods. In this paper we take a fresh look at fair division, by approaching it from the mechanism design point of view, and in particular in search of truthful partition mechanisms.

1.1 Background

The problem of fair division, or cake cutting, is a central and classical problem in economics. A *fair* partition is one where each of k players receives at least $1/k$ of the cake, each according to her own measure (this is also known as a *proportional* partition). Fair partition into two parts has long been known to be possible using a "cut-and-choose" procedure: a cake is guaranteed to be fairly divided in two if one player cuts it and the other picks a piece.

Fair partition to more than two players requires non-trivial mathematics. The founding work in this field was done by Steinhaus, Knaster and Banach in the forties (e.g. [7], [12], [13]). They and others (e.g. [10]) proved existence theorems for fair division on general measure spaces and

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non-atomic measures, where "fairness" was again, in general, taken to mean that each of k players receives at least $1/k$ of the cake, each according to her own measure. These proofs are not constructive, but nevertheless were useful in generalizing "cut-and-choose" to more than two players.

Another natural approach is "moving knife algorithms", first described by Dubins and Spanier [4]. These are not algorithms in the modern Turing Machine sense of the word (as "cut-and-choose" and its generalizations aren't), but still provide a "practical" way to cut actual cakes, if nothing else, by the same fairness criterion mentioned above.

A stronger concept of fairness is that of "envy-free" partitions ([6],[14]). In an envy free partition there is no player who would trade her allocation with one given to another player. Such partitions have been studied extensively (e.g., see [1]).

Partitioning indivisible goods is a more recently studied variant, which places the problem in a standard algorithmic setup. However, it is easy to see that fairness or envy-freeness could not be achieved in every setup as the example of one good demonstrates. Still, the results of Lipton et al. [8] show that almost envy free partitions exist. These are partitions where no player envies another player by more than the value of a single good.

Classical cake cutting mechanisms such as "cut-and-choose" usually require the players to choose a piece of a subset of the cake or make a cut in it, according to their preferences. Alternatively, the mechanism queries the player about her valuation of a particular piece of the cake. We take an approach which is more prevalent in the modern mechanism design world: each player declares their preference (i.e., their entire measure on the cake) to some "third party", which then proceeds to divide the cake according to a predetermined algorithm.

While circulating a draft of this paper, it was brought to our attention that similar questions are discusses in a recent working paper by Chen et al. [3]. They restrict themselves to a particular class of measures: "the case where the agents hold piecewise uniform valuation functions, that is, each agent is interested in a collection of subintervals of $[0,1]$ with the same marginal value for each fractional piece in each subinterval." In this setting, they show a truthful, deterministic, polynomial-time, fair and envy-free mechanism. For the more general setting which we study they independently derive our mechanism 2.1.

1.1.1 Truthfulness

The notion of truthfulness is very natural in partitioning problems. Why would a player declare her true measure if declaring a different measure would result in a better partition for her?

The "cut-and-choose" method is in some sense truthful: the players don't have to trust each other to be guaranteed a half of the cake. However, it is *not* truthful in the sense that players have incentive to "strategize" in order to increase the value of the piece they receive. For example, assume that the players' preferences differ, and that the first player (the "cutter") knows, or guesses, the second player's (the "chooser") preference. She may then cut the cake into two pieces such that the first piece is worth much more than one half to herself, and $\frac{1}{2} - \epsilon$ to the chooser. This would, perhaps, secure her a large piece of the cake, and leave the chooser with the feeling that dealings were not completely *fair*. To further confound matters we note that the chooser may realize, in this setting, that she could gain a larger piece by manipulating the cutter's perception of her preferences.

As noted in [1], many of the partition mechanisms which were discovered in the non-atomic case have this same weaker truthfulness property; for fair partitions - they have the property that truthful players are guaranteed to receive at least $1/k$ of their total value of the cake. Similarly for envy free partitions every truthful player is guaranteed not to envy any of the other players.

Still the question remains as to why is it beneficial for a player to declare her true value?

A work which addresses this question is a paper by Lipton et al. [8] who analyze a truthful mechanism for allocating a set of indivisible goods: simply give each good to each of k players with probability $1/k$. This mechanism is further analyzed in [2], who showed that this partition results, with high probability, in $O(\alpha\sqrt{n\ln k})$ envy, where n is the number of goods, k is the number of players, and α is the maximum utility over all goods and players.

The strongest possible notion of truthfulness is the following: a mechanism is *truthful* if it is always the case that a player's utility when declaring the truth is as high as her utility when lying. The notion of [8, 2] is weaker. They focus, as we do, on *truthfulness in expectation*: the expected value of a player's utility is maximal when telling the truth.

1.2 Our Results

1.2.1 Non Atomic Measures

Although our "third party" approach is one of modern mechanism design, our work in the non-atomic setup lies in the realm of classical fair division, in that it does not consider the computational aspects of the mechanisms: in some cases, an infinite amount of information would have to be conveyed to the third party. Even when not, the calculations required may be intractable or even not recursive.

We first consider the problem of a truthful fair partition with non-atomic measures. For this problem, using a result of Steinhaus [13], we show that such a partition always exists. Furthermore, our mechanism particularly incentivizes risk averse players to play truthfully.

We next ask if it is possible to devise a truthful mechanism that guarantees that each player gets more than $1/k$ of the cake. Obviously the answer is negative if all the measures are the same. However, as mentioned by Steinhaus [12] and proved by Dubins and Spanier [4], if the measures are not all identical, then there exists a partition where each player gets more than $1/k$ of the entire cake.

Our main result is a *truthful in expectation* mechanism which guarantees that each player gets at least $1/k$ of the cake, and furthermore the expected size of the piece each player gets is strictly larger than $1/k$.

We further show that there exists no deterministic and truthful mechanism that gives these guarantees and so only in the weaker notion of truthfulness it is possible to obtain such results.

An additional argument in favor of randomized mechanisms is that a deterministic mechanism cannot be symmetric: when all the players declare the same preferences, it must arbitrarily break the symmetry and assign different players different slices of the cake. A randomized algorithm can avoid this.

1.2.2 Indivisible Goods

Our results presented in the case of non-atomic measures are existential; we do not provide protocols for implementing them. To address the computational aspect of the problem we consider partitions of a large number of indivisible goods where the number of goods n is exponential in the number of players, and the utility of a single good is bounded. We give efficient versions of all of our mechanisms in this setup, with guarantees that are slightly weaker than those provided in the continuous case. The guarantees hold in expectation, and moreover there is a deterministic guarantee for each player to receive a share of the goods which is at least $(1-\epsilon)$ of the expected value, where $\epsilon = O(M^k/n)$ where k is the number of players, n is the number of goods and M is the maximum utility of a single good.

More generally, we prove a discrete analogue to a theorem of Dubins and Spanier [4]. They show that the space of partition utilities is convex. We show that the same is true in the discrete case, again up to a factor of $(1 - \epsilon)$, as defined above.

2 Continuous Truthful Mechanisms

2.1 Existence Theorems for Fair Divisions

Dubins and Spanier [4], rephrasing Fisher [5], provide the following description of what they call "The problem of the Nile":

"Each year the Nile would flood, thereby irrigating or perhaps devastating parts of the agricultural land of a predynastic Egyptian village. The value of different portions of the land would depend upon the height of the flood. In question was the possibility of giving to each of the k residents a piece of land whose value would be $1/k$ of the total land value no matter what the height of the flood."

Neyman [10] showed that this is possible, given that there are a finite number of levels that the Nile can rise to.

Let $\mathcal{C}$ be a "cake" (a set), and $\mathfrak{C}$ a set of "slices" (a σ-algebra of subsets of $\mathcal{C}$). Let there be k players, and let $\mu_1, \ldots, \mu_k$ be non-atomic probability (additive) measures on $(\mathcal{C}, \mathfrak{C})$, so that the value of a slice $C \in \mathfrak{C}$ to player i is $\mu_i(C)$. Then Neyman's theorem establishes that there exists a partition of the cake $C_1, \ldots, C_k$ such that for all players i and slices j it holds that $\mu_i(C_j) = 1/k$. Hence all the slices are equal, by all the player's measures.

Dubins and Spanier [4] show that a better partition is always possible when at least two of the players have different measures. Their theorem implies that in this case a partition is possible for which, for all players and slices i, it holds that $\mu_i(C_i) > 1/k$, and so each player gets strictly more than $1/k$ of the cake, by his or her own measure.

2.2 Truthful Mechanisms

2.2.1 Fair division

We present a simple truthful "mechanism" for distributing the cake among k players, which assures that each player gets precisely $1/k$ of the cake, by all the players' measures. It is a "mechanism" in quotes since it is as constructive as Neyman's theorem, which is not constructive. Note that this mechanism also appears in [3].

Mechanism 2.1. *Assume the players' true measures are $\mu_1, \ldots, \mu_k$, and that they each declare some measure ν_i. Find a partition $C_1, \ldots, C_k$ such that $\forall i, j : \nu_i(C_j) = 1/k$. Then choose a random permutation τ of size k, from the uniform distribution, and give $C_{\tau(i)}$ to player i.*

Proposition 2.2. *Mechanism 2.1 is* truthful *in the following sense: No player can increase her expected utility by playing non-truthfully. Further, a player who plays truthfully minimizes the risk/variance of the measure of the piece she gets.*

Proof. The expected size of the slice for player i is $\sum_j \mu_i(C_j) \mathbf{P}[\tau(i) = j] = \sum_j \mu_i(C_j)/k = \mu_i(\cup C_j)/k = 1/k$. Since it is independent of ν_i then player i has no incentive to declare untruthfully. Furthermore, a player that declares $\nu_i = \mu_i$ is guaranteed a slice of size $1/k$, and so the truth minimizes the variance (or risk), to zero. $\square$

2.2.2 Super-fair division

For this result we set $\mathcal{C} = [0,1) \in \mathbb{R}$ and let $\mathfrak{C}$ be the Borel σ-algebra. While this result can be extended to more general classes of spaces and algebras, we present it in this restricted form for clarity. We consider the case where at least one pair of measures are not equal, i.e. the case in which "super-fair" partitions exist — partitions in which $\mu_i(C_i) > 1/k$.

We first provide motivation for our usage of "truthfulness in expectation", by showing that deterministic "super-fair" mechanisms cannot be truthful:

Theorem 2.3. *Any deterministic mechanism that gives each player $1/k$ of the cake when all declared measures are equal and more than $1/k$ of the cake otherwise is not truthful.*

Proof. Consider the case where all players declare the same arbitrary measure μ. Then they receive slices C_i such that $\forall i : \mu(C_i) = 1/k$. Now, consider the case where player 1's true measure ν is such that $\nu(C_1) = 1$. Then player 1's utility for declaring μ is 1. We propose that her utility for declaring ν (i.e., being truthful) is less than one, and therefore the mechanism is not truthful.

Assume by way of contradiction that her utility for declaring ν is one. Then in this case she must also receive slice C_1, since that is the only slice worth one to her. But if player 1 receives C_1 then the rest of the players have, by their measure μ, exactly $(k-1)/k$ of the cake left to share, and so it is impossible that they all receive more than $1/k$ of it. This contradicts the hypothesis, since $\mu(C_1) = 1/k$ and $\nu(C_1) = 1$, and therefore $\mu \neq \nu$ and all players must receive more than $1/k$. $\square$

We now describe a randomized mechanism that is "super-fair" and truthful in expectation.

Mechanism 2.4. *Assume again that $\mu_1, \ldots, \mu_k$ are the players' true measures, and that they each declare some measure ν_i. To distribute the cake, pick a partition $C_1, \ldots, C_k$ from a distribution D over partitions, which we describe below. If it so happens that $\nu_i(C_i) > 1/k$ for all i, then distribute the slices accordingly. Otherwise distribute by mechanism 2.1, that is, give a slice of value $1/k$ to all players.*

Proposition 2.5. *In mechanism 2.4 the expected utility of a truthful player is larger than $1/k$ if super-fair partitions are picked with positive probability.*

Proof. If $C_1, \ldots, C_k$ is super-fair and the players are truthful, then this partition is recognized as super-fair, and the players each get strictly more than $1/k$. In the event that the picked partition is not super-fair, and the players are truthful, then the mechanism reverts to giving the players precisely $1/k$ of the cake. Thus the expectation for truthful players is more than $1/k$. $\square$

Proposition 2.6. *In mechanism 2.4 playing truthfully maximizes a player's expected utility.*

Proof. Consider again two cases: the first, in which $C_1, \ldots, C_k$ is super-fair, and the second, in which it isn't.

In the first event, a truthful player's expected share is more than $1/k$. Playing untruthfully could either have no effect, leaving the utility as is, or else the only other possibility is that the partition is misconstrued not to be super-fair, in which case the player's utility is reduced to $1/k$.

In the second event, in which the partition picked is not super-fair, playing untruthfully may again have no effect, leaving the utility at $1/k$. However, if, to some player, the share allocated by this partition was worth less than $1/k$, playing untruthfully may make it seem to be valued at more then $1/k$, turning the partition into super-fair by the declared preferences, and resulting in a utility less than $1/k$ for that player.

Thus, for any random choice of $C_1, \ldots, C_k$ the truthful player's expected utility is maximal, and the proposition follows. $\square$

To assure that this mechanism results in a slice of expected size strictly greater than $1/k$, we must find a distribution D (from which we draw the partition) such that for any set of measures, where at least one pair is not equal, with positive probability $\mu_i(C_i) > 1/k$ for all i. To this end we make the following definition:

Definition 2.7. *Denote by $\mathcal{Q}$ the set of partitions $C_1, \ldots, C_k$ of $[0, 1) \in \mathbb{R}$ for which each C_i is a finite union of half-open intervals with rational endpoints.*

We note that $\mathcal{Q}$ is countable. D now need only be some distribution with support $\mathcal{Q}$:

Theorem 2.8. *Let $\mu_1, \ldots, \mu_k$ be non-atomic probability measures on $[0, 1) \in \mathbb{R}$ and the Borel σ-algebra, such that there exist i, j for which $\mu_i \neq \mu_j$. Let D be a distribution over the partitions $C_1, \ldots, C_k$ of $[0, 1)$ into k sets, such that the support of D is $\mathcal{Q}$. Then*

$$\mathbf{P}_D[\forall i : \ \mu_i(C_i) > 1/k] > 0.$$

The proof appears in Appendix A.

3 Indivisible Goods

Let $\mathcal{C} = \{a_1, \ldots, a_n\}$ be a finite set of indivisible goods ("discrete cake"). Let there be k players, where each has an additive bounded measure (utility) on the algebra of subsets of $\mathcal{C}$, μ_i, such that for all i, j it holds that $\mu_i(\{a_j\}) \in \{1, 2, \ldots, M\}$.

We focus on the regime where the number of players is small, so that $n \gg M^k$, and in particular $Mk \cdot M^k/n < \epsilon$ for some ϵ. Then it also holds that $Mk \cdot M^k/\mu_i(\mathcal{C}) < \epsilon$.

3.1 Truthful fair division

Let $\nu_1, \ldots, \nu_k$ be the set of declared measures.

Definition 3.1. *Let $S = (s_1, \ldots, s_k) \in \{1, 2, \ldots, M\}^k$ be some vector. Let the bin $B_S \subseteq \mathcal{C}$ be the set of goods a for which, for each player i, it holds that $a \in B_S$ iff $\nu_i(a) = s_i$:*

$$B_S = \{a \in \mathcal{C} \ s.t. \ \forall i : \nu_i(a) = s_i\} \tag{1}$$

Let $\mathcal{B}$ be the set of bins.

We propose the following mechanism:

Mechanism 3.2. *For each bin B_S, pick from the uniform distribution a partition of it into k parts of equal size $B_S^1, \ldots, B_S^k$, with perhaps some left over elements which number at most $k - 1$. Let $C_i' = \bigcup_{B_S \in \mathcal{B}} B_S^i$ and give C_i' to player i. Then, give each leftover good to some player, uniformly at random.*

Denote by C_i the set that player i got, i.e. C_i' union any leftovers given to player i. Then it is easy to see that this mechanism is truthful, since player i's expectation is $\mu_i(\mathcal{C})/k$, independently of her declared measure ν_i:

Proposition 3.3.
$$\mathbf{E}[\mu_i(C_j)] = \mu_i(\mathcal{C})/k$$

Proof. This follows from the fact that every a_l ends up in C_j with probability $1/k$. $\qquad \square$

Truthfulness, however, could have been more simply achieved by, for example, giving each player the entire set $\mathcal{C}$ with probability $1/k$. This mechanism's merit is that it ensures low risk for truthful players:

Theorem 3.4. *When $\nu_i = \mu_i$ then for all j it holds that $\mu_i(C_j) \geq (1 - \epsilon)\mu_i(\mathcal{C})/k$.*

when $i = j$ this implies low risk for truthful players.

Proof. By definition of the C_j's

$$\mu_i(C_j) \geq \mu_i(C_j') \quad = \quad \mu_i\left(\bigcup_{B_S \in \mathcal{B}} B_S^j\right).$$

Since the different B_S^i's are disjoint, and by the definition of B_S

$$\mu_i(C_j) \geq \sum_{B_S \in \mathcal{B}} \mu_i(B_S^j) \geq \sum_{B_S \in \mathcal{B}} s_i |B_S^j|.$$

We denote the number of left over elements r_S, so that $|B_S| = k|B_S^i| + r_S$ for all i. Then

$$\mu_i(C_j) \quad \geq \quad \frac{\mu_i(\mathcal{C})}{k} - \frac{1}{k}\sum_{B_S \in \mathcal{B}} s_i r_S,$$

since $r_S < k$ and $s_i \leq M$, and by the definition of ϵ we finally have that

$$\mu_i(C_j) \geq \frac{\mu_i(\mathcal{C})}{k} - \frac{Mk \cdot M^k}{k} \geq (1 - \epsilon)\mu_i(\mathcal{C})/k. \tag{2}$$

$\square$

We conclude that assuming players are averse to risk, they may find actual advantage in playing truthfully, since that will result in a utility that is with probability one greater than $(1 - \epsilon)\mu_i(\mathcal{C})/k$. Other strategies, on the other hand, may run the risk of resulting in lower utility.

3.2 Truthful super-fair division

We can naturally adapt mechanism 2.4 to the discrete case, by letting D be the uniform (for example) distribution over the partitions of $\mathcal{C} = \{a_1, \ldots, a_n\}$ into k subsets. We then use what is essentially the same mechanism:

Mechanism 3.5. *Pick a random partition from D, keep it iff everyone was allocated strictly more than $1/k$, and otherwise give everyone $1/k$ using mechanism 3.2.*

In this discrete case it is easy to see that if super-fair divisions exist then they are picked with positive probability. The proofs that this mechanism results in expected utility larger than $1/k$, and that it is truthful, are identical to the ones for the continuous case, 2.5 and 2.6.

3.3 Extending continuous fair division existence results

Let $\mathcal{M}$ be the space of k-by-n matrices M such that for measures $\mu_1, \ldots, \mu_k$ and some division $C_1, \ldots, C_n$ it holds that $M_{ij} = \mu_i(C_j)$. Dubins and Spanier [4], using a theorem of Lyapunov [9], prove that $\mathcal{M}$ is compact and convex when the measures μ_i are non-atomic. From this follow a plethora of existence theorems for partitions of different characteristics. For example, as mentioned above, this fact can be used to show that there exists a division where each of the k players gets a share worth $1/k$ by everyone's measure (for probability measures). It can also be used to show that some players have different measures then a division exists in which every player gets more than $1/k$, by her own measure.

This result obviously does not apply to the discrete case; the set of partitions is finite and it is difficult to speak of convexity. Accordingly, in the general case no fair partition exists, and a super-fair partition may not exist even when the preferences are different. A simple example of two goods and three players suffices to illustrate this point.

One could, however, imagine that all this *could* be achieved if the players were somehow able to share the goods. In fact, if we allow, for example, that one player has a third of a good and another two thirds of it (with appropriate utilities), then Dubins and Spanier's results apply again, and a wealth of partitions with different qualities is possible again. We refer to this as the continuous extension of the discrete problem.

However, indivisible goods must by nature remain indivisible. To overcome this, we propose a randomized partition, similar to the one used in mechanism 3.2, that makes possible, *in expectation*, any partition values possible in the continuous extension. Moreover, it assures that each player not only receives the correct utility in expectation, but that in the worst case she will not receive less than $1 - \epsilon$ of what she expects.

We thus consider again a set of indivisible goods $\mathcal{C} = \{a_1, \ldots, a_l\}$, k players, and their additive measures μ_i, where $\mu_i(\{a_j\}) \in \{1, 2, \ldots, M\}$. We now imagine that each good can be continuously subdivided, and so define a_j^* to be copy of $[0, 1] \in \mathbb{R}$, and let $\mathcal{C}^* = \{a_1^*, \ldots, a_l^*\}$. Let $\mathfrak{F}$ be the standard Borel σ-algebra on $\mathcal{C}^*$, and let ν be the Lebesgue measure on $\mathfrak{F}$. Define μ_i^*, a measure on $\mathfrak{F}$, as a continuous extension of μ_i by

$$\mu_i^*(A) = \sum_j \mu_i(\{a_j\}) \cdot \nu(A \cap a_j^*),$$

for any $A \in \mathfrak{F}$. We refer to $\mathcal{C}^*$ and μ_i^* as the *continuous extension* of $\mathcal{C}$ and μ_i.

We are now ready to state the main result of this section:

Theorem 3.6. *Consider the problem of partitioning indivisible goods as defined above, and its continuous extension. Let $\mathcal{M}$ be the space of k-by-n matrices M such that for some $C_1^*, \ldots, C_n^*$ it holds that $M_{ij} = \mu_i^*(C_j^*)$.*

Then for every element $M \in \mathcal{M}$ there exists a randomized partition $C_1, \ldots, C_n$ satisfying $\mathbf{E}[\mu_i(C_j)] = M_{ij}$, and moreover

$$\frac{\mu_i^*(C_j^*)}{\mu_i^*(\mathcal{C}^*)} + k\epsilon \geq \frac{\mu_i(C_j)}{\mu_i^*(\mathcal{C}^*)} \geq \frac{\mu_i^*(C_j^*)}{\mu_i^*(\mathcal{C}^*)} - \epsilon,$$

where ϵ, as before, is $O(M^k/n)$.

Proof. Given a division $C_1^*, \ldots, C_n^*$ of the divisible C^*, we would like to divide the discrete $\mathcal{C}$ in a way that approximates this division as closely as possible. That is, we would like to find a division $C_1, \ldots, C_n$ such that $\mu_i(C_j) \approx \mu_i^*(C_j^*)$. We propose two schemes to do this: the random scheme and the binned scheme. For both of them, we define an n-by-l matrix D (l being the number of indivisible goods), where D_{ij} is the fraction of good a_j^* that belongs to C_i^*: $D_{ij} = \nu(C_i^* \cap a_j^*)$.

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The random scheme. In the random scheme, we simply give player i good a_j with probability D_{ij} (note that by the definition of D, $D_{i\cdot}$ is a distribution). Then

$$\mathbf{E}[\mu_i(C_j)] \;\; = \;\; \sum_m \mu_i(\{a_m\}) \cdot \mathbf{P}[a_m \in C_j] = \sum_m \mu_i(\{a_m\}) \cdot D_{jm} = \mu_i^*(C_j^*)$$

and its standard deviation is $O\left(\sqrt{\mu_i^*(C_j^*)}\right)$.

The binned scheme. In the binned scheme, we bin the elements of $\mathcal{C}$ into bins $\{B_S\}$ as above. Without loss of generality, let there be, for each player i and bin B_S, a single value D_{iS} such that $D_{im} = D_{iS}$ for all m. No generality is indeed lost: because all elements of a bin are equivalent to all the players, then for any partition C_j^* there exists an equivalent partition, in the sense of the utilities of the players, for which such D_{iS}'s exist.

Let n_S be the number of elements in bin B_S. From each bin B_S, we give player i a number of goods equal to $\lfloor n_S D_{iS} \rfloor$, picked from the uniform distribution over such partitions. Any leftover a_m we give according to the random scheme, i.e. to player i with probability D_{iS}.

The expectation for $\mu_i(C_j)$ clearly remains $\mu_i^*(C_j^*)$. However, here we can bound it from below:

$$\frac{\mu_i(C_j)}{\mu_i^*(\mathcal{C}^*)} \geq \frac{1}{\mu_i^*(\mathcal{C}^*)} \sum_{B_S \in \mathcal{B}} (n_S D_{iS} - 1) s_i \geq \frac{\mu_i^*(C_j^*)}{\mu_i^*(\mathcal{C}^*)} - \epsilon.$$

We can also bound it from above, by noting that the most a player can get beyond $\mu_i(C_j)$ is what's lost by the rest of the players:

$$\frac{\mu_i^*(C_j^*)}{\mu_i^*(\mathcal{C}^*)} + k\epsilon \geq \frac{\mu_i(C_j)}{\mu_i^*(\mathcal{C}^*)}$$

$\square$

A Existence of a distribution on partitions with positive probability for super-fair division

Let $\mathcal{R}$ be the set of finite unions of half-open intervals of $[0, 1) \in \mathbb{R}$ with rational endpoints. Let $\mathcal{Q}$ be defined to be the class of partitions of $[0, 1) \in \mathbb{R}$ such that each part of the partition is in $\mathcal{R}$.

The main lemma we want to prove is the following:

Lemma A.1. *Let $\mu_1, \ldots, \mu_k$ be non-atomic Borel measures on $[0, 1) \in \mathbb{R}$ and the Borel σ-algebra $\mathcal{B}$. Let $B_1, \ldots, B_k$ be a partition of the interval, where $B_i \in \mathcal{B}$ and $\delta > 0$. Then there exists a partition $Q_1, \ldots, Q_k$ in $\mathcal{Q}$ such that $\mu_i(B_i \triangle Q_i) < \delta$ for all i.*

Proof. The proof uses the fact that $\mathcal{Q}$ is an algebra, i.e., it is closed under finite unions, intersections and taking of complements. Let δ' be chosen later. Since all of the measures are Borel we can find open sets $O_{i,j}$ so that $B_j \subset O_{i,j}$ and $\mu_i(O_{i,j} \triangle B_j) < \delta'$ (see, e.g., theorem 2.17 in [11]). Taking $O_j = \cap_i O_{i,j}$, we get open sets such that $B_j \subset O_j$ and $\mu_i(B_j \triangle O_j) < \delta'$ for all i and j.

Fix j and note that $O_j = \cup_{n=1}^{\infty} I_{j,n}$, where $I_{j,n}$ are open intervals with rational end-points. Take m sufficiently large so that $\mu_i(O_j \setminus \cup_{n=1}^{m} I_{j,n}) < \delta'$ for all i. Let $J_{j,n}$ be the same as $I_{j,n}$ except that the left-end point of the interval is added and let $\tilde{P}_j = \cup_{n=1}^{m} J_{j,n}$. Since the measures are non-atomic we have $\mu_i(\cup_{n=1}^{m} J_{j,n}) > \mu_i(O_j) - \delta'$. Note that the $\tilde{P}_j$'s are all unions of half-open intervals with rational end-points. Moreover, for all i and j,

$$\mu_i(\tilde{P}_j \triangle B_j) \leq \mu_i(\tilde{P}_j \triangle O_j) + \mu_i(O_j \triangle B_j) \leq 2\delta'.$$

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Note further that for all i it holds that:

$$\mu_i[0,1) \geq \mu_i(\cup_j \tilde{P}_j) \geq \mu_i(\cup_j O_j) - \sum_j \mu_i(O_j \setminus \cup_{n=1}^m I_{j,n}) \geq \mu_i[0,1) - k\delta',$$

so

$$\mu_i([0,1) \setminus \cup_j \tilde{P}_j) \leq k\delta'.$$

Now take $P_i = \tilde{P}_i$ for $i > 1$ and $P_1 = \tilde{P}_1 \cup ([0,1) \setminus \cup_j \tilde{P}_j)$. Then $\cup P_i = [0,1)$ and

$$\mu_i(P_j \triangle B_j) \leq (k+2)\delta'$$

for all i and j. The P_i are almost the desired partition. They satisfy all the needed properties except that they are not a partition. We now take $Q_j = P_j \setminus \cup_{j' < j} P_{j'}$. Q_j is obviously a partition. Moreover:

$$
\begin{aligned}
\mu_i(Q_j \triangle B_j) &\leq \mu_i(P_j \triangle B_j) + \sum_{j' \neq j} \mu_i(P_{j'} \triangle B_j) \\
&\leq \mu_i(P_j \triangle B_j) + \sum_{j' \neq j} \mu_i(B_{j'} \triangle B_j) + \sum_{j' \neq j} \mu_i(P_{j'} \triangle B_{j'}) \\
&\leq 2k(k+2)\delta'.
\end{aligned}
$$

Taking $\delta' = \delta/(2k(k+2))$ concludes the proof. $\qquad\square$

Theorem A.2. *Let $\mu_1, \ldots, \mu_k$ be non-atomic probability measures on $[0,1) \in \mathbb{R}$ and the Borel σ-algebra, such that there exist i,j for which $\mu_i \neq \mu_j$. Let D be a distribution over the partitions $C_1, \ldots, C_k$ of $[0,1)$ into k sets, such that the support of D is $\mathcal{Q}$. Then*

$$\mathbf{P}_D[\forall i : \mu_i(C_i) > 1/k] > 0.$$

Proof. By Dubins and Spanier's theorem, there exists a partition $B_1, \ldots, B_k$ of measurable sets such that for all i it holds that $\mu_i(B_i) > 1/k$. Let $\epsilon > 0$ be such that $\mu_i(B_i) > 1/k + \epsilon$.

By the lemma above there exists a partition $Q_1, \ldots, Q_k$ in $\mathcal{Q}$ such that $\forall i,j : \mu_i(Q_j \triangle B_j) < \frac{1}{2}\epsilon$. This, in particular, implies for all i that $\mu_i(Q_i) > 1/k + \frac{1}{2}\epsilon$. $\qquad\square$

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